Deriving Moments in Age of Gossip from Percolation

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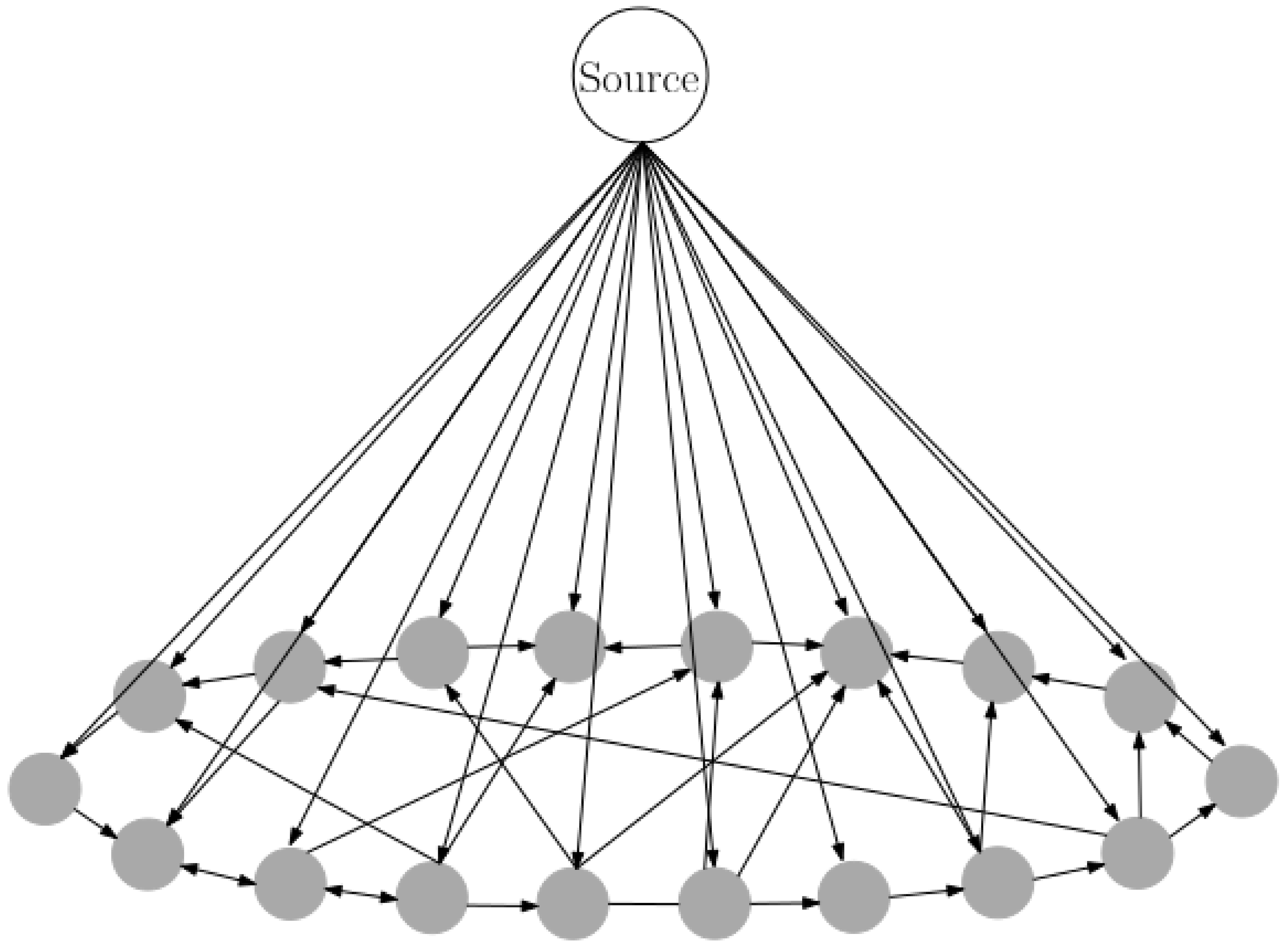
Introduction

- We consider a network where a source generates packets and forwards them to a network containing n nodes.
- The nodes in the network send updates to neighbors using the randomized **gossip protocol**.
- We recover existing equalities for the Age of Information (AoI) metric by leveraging a recent connection to **First-Passage Percolation (FPP)**.

System Model

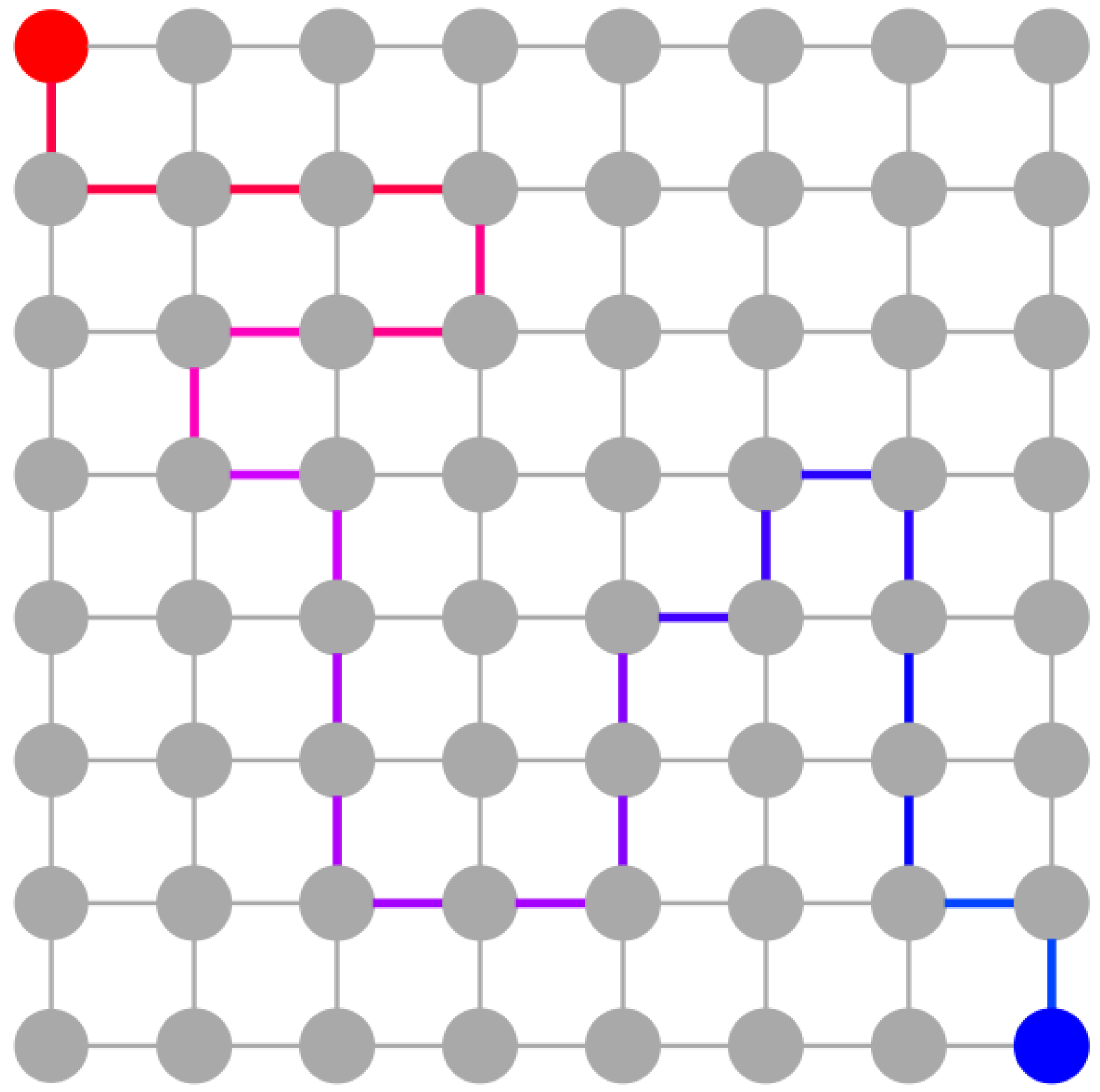


A gossiping network following the gossip protocol. Each node forwards packets to neighboring nodes. Arrows denote the flow of information, with the source only sending updates and nodes in the network both sending and recieving updates.

Objectives

- Finalize the connection between First-Passage Percolation (FPP) and Age in Gossip Networks
  - Connection first noted by *Maranzatto and Michelen (2025)*
- Use FPP instead of control-theoretic tools to exactly characterize the moments of the Age of Gossip.
- Strengthen existing formula to hold in infinite graph, finite time setting (previous results only held for finite graphs in the large-time limit)
- New connections between AoI in gossip and classical models in Statistical Physics (Eden Model)

First Passage Percolation



Sample first passage path on a grid graph from the red vertex to the blue vertex. One can view the first passage time as the time a packet would take to travel from the red vertex to the blue vertex, along a minimizing random path.

Notation and Model

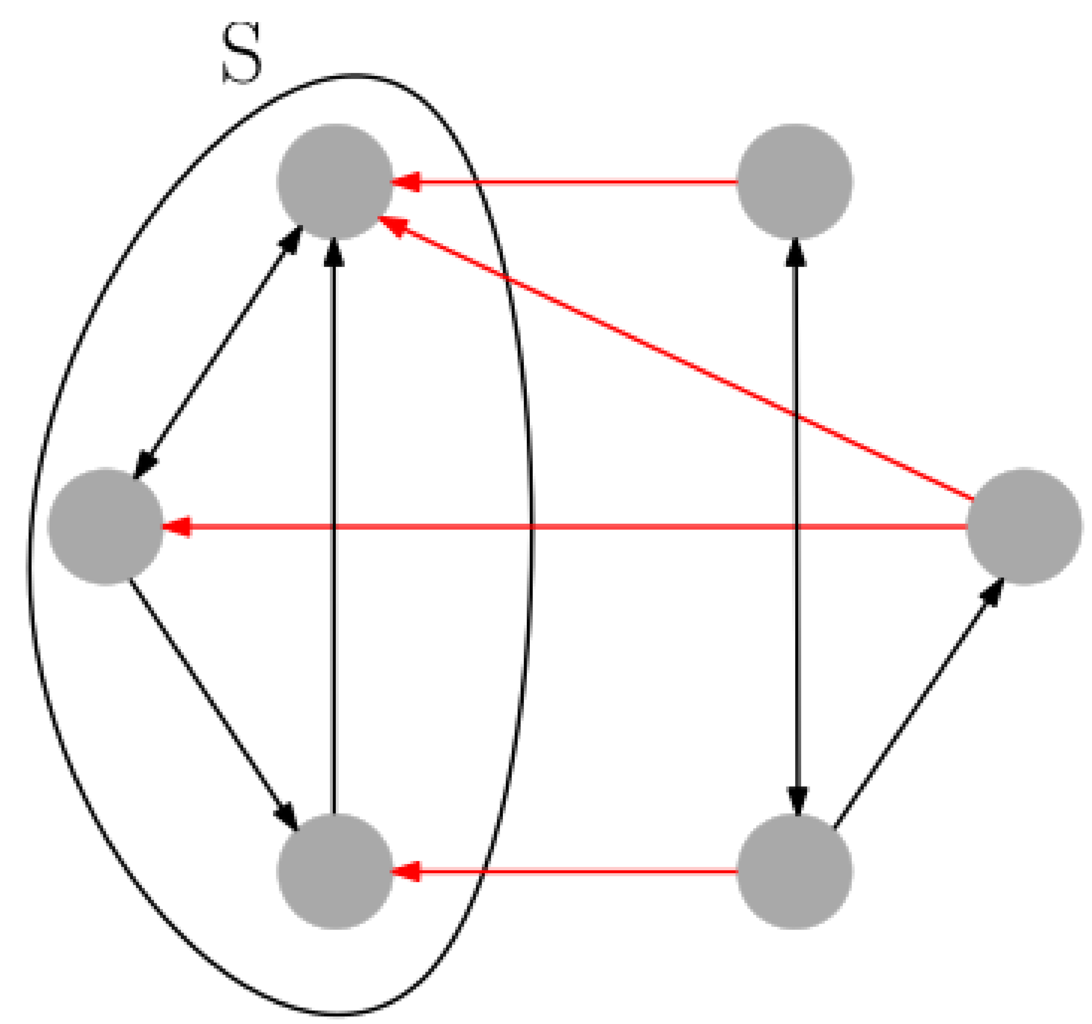
- Graph $G = (\mathcal{N}, E)$ with source node n_0.
- n_0 sends packets to G isotropically with rate λ.
- $u \to v$ has an associated Poisson Process with rate λ_{uv}
- The total rate into S is $\sum_{e \in E(s)} \lambda_e$
- When u updates v, node v takes the fresher packet
- $N_v(t)$ is timestamp of current packet at node u
- The AoI of v is then $X_v(t) \triangleq t - N_v(t)$
- The First-Passage Time $T_G(n_0, v)$ is the shortest path from n_0 to v in G, if all Poisson Processes are replaced by corresponding $\mathrm{Exp}(\lambda_{uv})$ variables.

Main Result: Strengthened Recursions

For any **locally finite** G, any subset $S \subset G$ and large enough **finite time** t, the kth AoI moment of S satisfies,

$$\mathbb{E}[X_S^k] = \frac{k\mathbb{E}[X_S^{k-1}] + \sum_{e \in E(S)} \lambda_e \mathbb{E}[X_{S\cup e}^k]}{\lambda_0(S) + \sum_{e \in E(S)} \lambda_e}. \tag{1}$$

Subsets for Recursion



In the proofs of our main results, the random variable Y is one of those edges in $E(S)$, highlighted in red above. The black edges are not included in $E(S)$. We have not included the source vertex n_0, but implicitly the edges from n_0 to the network are still present.

Derivation for First Moment

For any subset S, let Y be the random variable indexing the next edge that rings in $E(S)$. Let $D = \lambda_{n_0}(S) + \sum_{e \in E(S)} \lambda_e$. We have the following equalities,

$$\mathbb{P}[Y = e] = \frac{\lambda_e}{D}$$

$$\mathbb{E}[X_S | Y = e] = \begin{cases} \mathbb{E}[\mathrm{Exp}(D)], & n_0 \in e, \\ \mathbb{E}[X_{S\cup e} + \mathrm{Exp}(D)], & n_0 \notin e. \end{cases}$$

By the law of total expectation we have,

$$\begin{aligned}
\mathbb{E}[X_S] &= \mathbb{E}[\mathbb{E}[X_S|Y]] \\
&= \sum_{e \in E(S)} \mathbb{E}[X_S|Y=e]\mathbb{P}[Y=e] \\
&= \frac{1}{D}\left(\sum_{\substack{e \in E(S) \\ n_0 \in e}} \lambda_e \mathbb{E}[\mathrm{Exp}(D)] + \sum_{\substack{e \in E(S) \\ n_0 \notin e}} \lambda_e \mathbb{E}[X_{S\cup e} + \mathrm{Exp}(D)] \right) \\
&= \frac{1}{D}\left(\frac{\lambda_0(S)}{D} + \sum_{\substack{e \in E(S) \\ n_0 \notin e}} \lambda_e \mathbb{E}[X_{S\cup e}] + \frac{\lambda_e}{D} \right) \\
&= \frac{1}{D}\left(1 + \sum_{\substack{e \in E(S) \\ n_0 \notin e}} \lambda_e \mathbb{E} X_{S\cup e} \right) \quad \square
\end{aligned}$$